ABC conjecture in the school XXI. Prime numbers as the metalanguage of the Universe

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Аннотация

The ABC conjecture states that for three mutually prime numbers A, B, and C satisfying the relation A + B = C, the product of the prime divisors A, B, and C is usually not much less than C. The ABC conjecture is very simple to formulate, but extremely difficult to prove. Some five hundred pages have been spent by some of the most eminent mathematicians in the Western world trying to find a proof - but the result is far from clear and difficult to verify. Meanwhile, any student with advanced training in the exact sciences can understand and prove the ABC conjecture, relying on creative imagination based on the synthesis of school knowledge, including physics and chemistry. The ABC hypothesis has a number of remarkable consequences in number theory, such as the proof of Fermat's Last Theorem. Prime numbers have a huge research potential, they play a connecting role between the world around us and the microcosm. Prime numbers are therefore the foundation of the natural sciences. In this paper, the ABC hypothesis is proved mainly on the basis of a school curriculum with enhanced physical and mathematical training. This makes the proof easy to verify and is important for pedagogy.

Keywords: ABC-conjecture, entropy, physics, symmetry, pedagogy, universe

Contents

1 Introduction

1.1 Formulations of ABC conjecture

The ABC conjecture was formulated independently in number theory by the mathematicians David Masser in 1985 [1] and Joseph Esterle [2] in 1988. Further in the text: Oesterlé–Masser conjecture. It is stated in terms of three positive whole A,B C that are co-prime and satisfy A + B = C. The conjecture essentially states that the product of the distinct prime factors of ABC is usually not much smaller than C or $max(A, B, C) < K_\epsilon Rad(ABC)^{1+\epsilon}$, where K_ϵ depend only on some positive ϵ - real number.

The *Rad* function is the radical of the numbers A,B and C equal to the product of the primes forming these numbers, but raised to the first power, i.e. $Rad(8) = Rad(2^3) = 2$, $Rad(1000000) = Rad(2^6 * 5^6) = 30$. An equivalent formulation of the Oesterlé-Masser conjecture includes the quality q_{ABC} of the triple which is defined as:

$$q_{ABC} = \frac{log(C)}{logRad(ABC)} = \frac{ln(C)}{lnRad(ABC)}$$
(1.1)

For every positive real number ϵ, there exist only finitely many triples A, B,C, of co-prime positive integers with A + B = C such that $q_{ABC} > 1 + \epsilon$. Many hundreds of pages have been spent by eminent mathematicians in the Western world searching for proofs [3], and the process of searching for proofs continues.

1.2 The key idea of Prove

First of all, let's look at an example of the production of a batch of factory products. Let's assume that the law of distribution of measurable parameters of products - independent random variables A, B, C is the same and their mathematical expectations are respectively equal to $\overline{A}, \overline{B}, \overline{C}$ and are related by the ratio $\overline{A} + \overline{B} = \overline{C}$. These independent random variables have the same mean square deviation σ, the variance. From the basics of mathematical statistics we know that the variances of the random variables are summed and that the mean square deviation $\sim \frac{\sigma}{\sqrt{N}}$, where N - number of individual samples used in an experiment - *sample size*.

From this simple example we can proceed to the proof of the ABC conjecture. To do this, it is necessary to apply the basic theory of arithmetic, move on to simple integer polynomials and linear maps, Jordan matrices and quantum mechanics. The algebras of all the above are in homomorphism relations. The same prime numbers taken from the Rad(ABC) chain play the role of the roots of an integer polynomial, the characteristic polynomial of a linear mapping matrix, the root vectors of a linear space decomposed into a direct sum of invariant subspaces, and the eigenvalues for the Hamiltonian operator of a quantum system.

Concentrate on the eigenvalues/roots and make sure that the average of the logarithm of the matrices (*trace*, main diagonal) does not change due to multiple roots - this is the entropy S. Since entropy is the same additive quantity as the number of particles and both quantities are so-called integrals of motion, we have: relative standard deviation $\sim \frac{1}{\sqrt{S}}$, where S - entropy of the system under study. It remains to calculate $S = \frac{lnRad(ABC)}{L}$, where L is the length of the main diagonal (non-zero elements) of the matrix and at the same time the number of primes in the chain Rad(ABC). It corresponds to the statistical matrix of the system in Hilbert space, where the size of the matrix is infinite. When multiple roots appear, the probability of detecting a microobject in a given state is "smeared" along the main diagonal of the *statistical matrix* of

Table 1: *The general rule of homomorphism over algebras.*

Difficulty of understanding –>			
Prime Numbers	Integers polynomials	ODE Systems. Matrices	Quantum Mechanics (linear operators)
2.3.5.7.11 $\dots a = \prod_p p^{\alpha(p)}$	$f = c \prod_p p^{\alpha(p)}$	$\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y}$	$\widehat{H}\psi = E\psi$ and entropy formula $S = -\sum_i w_i ln(w_i)$
A prime to the power of k	is irreducible. polynomials $(\lambda - a_i)^k$	$R = R_1 \oplus R_2 \cdots \oplus R_i$ Invariant subspaces. The polynomials of matr.	Degenerate energy level.
The axioms of arithmetic	Possible algebraic operations	Description of dynamic processes	Models of the microcosm
<– Root Cause			

system in according to degree of the roots (degeneration of energy level). This proves the theorem and allows us to evaluate the so-called quality of the triple q, but in reality - a defect or fluctuation from above due to the restriction by a function with a normal distribution according to Chebyshev's theorem - we are talking about the entropy of the system under study. In short, even without knowing the function of the distribution of prime numbers $\pi(x)$, it is easy to make simple estimates. These estimates turn out to be sufficient to prove the ABC conjecture, the ABC hypothesis. The world is material, and the physics of the microcosm and the macrocosm are the same, the correspondence principle and the Heisenberg uncertainty principle work. So prime numbers model a system consisting of subsystems that do not <u>interact</u> but are <u>connected</u> by a common resource - *phase space*. For example, we can talk about a mixture of ideal gases It is difficult to overestimate the place of prime numbers in the physical laws that define the surrounding material world and, in a broad sense, our universe. See table (1).

It is the prime numbers that make it possi-ble to connect the system, the super-system and the sub-system.

2 The ABC conjecture from the Eye of Physicist

2.1 The Fundamental Theorem of Arithmetic

According to the basic theorem of arithmetic, each of the three numbers in the expression A + B = C can be decomposed into simple factors, powers above one are allowed, but under the conditions of the ABC conjecture, repeated factors in this decomposition are excluded due to the lack of common divisors.

$$A + B = C \Longleftrightarrow$$
$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k} * 1^{n-t} + b_1^{\beta_1} b_2^{\beta_2} \cdots b_l^{\beta_l} * 1^{n-u}$$
$$= c_1^{\gamma_1} c_2^{\gamma_2} \cdots c_m^{\gamma_m} * 1^{n-v} \quad (2.1)$$

In order to obey the rules of dimensionality of elements in the space of integers, defined as the n-fold Cartesian product of the axis of integers $Z^n = Z * Z * ... * Z$, each term

in the space of integers has to be a term in $Z^n = Z * Z * ... * Z$, each term in the above formula is also multiplied by a multiplier in the form of a 1^{n-t} elementary unit cube, so that each term has a dimension equal to the dimension of the Euclidean space n in which the manifolds of different dimensions under consideration are embedded. (From the analogy with the *Lego* constructor, you can try to move on to the axioms of topology and the postulates of Euclid). In other words, the sums of the exponents of all multipliers in the formula above are equal to t, u, v respectively, which corresponds exactly to the dimension of the manifolds A, B, C:

$$\alpha_1 + \cdots \alpha_k = t, \beta_1 + \cdots \beta_l = u, \gamma_1 + \cdot \gamma_m = v \tag{2.2}$$

It is easy to understand the dimension of each member and the minimum required dimension $n = max(t, u, v)$ (2.1) of the Euclidean space in which the studied cuboids can be embedded according to the given formula. The total number of simple factors of the above formula will be used below, it is equal to $L = k + l + m$.

Developing the technical approach adopted in the author's work [4], we compare the expression (2.1) with a set in the form of three manifolds, each in the form of a multidimensional parallelepiped or, in short, a *cuboid* in a finite-dimensional Euclidean space, with a given metric and measure, hereinafter referred to simply as Euclidean space. The homogeneity of Euclidean space is postulated).

$\exists$ a continuous invertible equivalence function f that maps each element as an elementary cube 1^n in Z^n of the sets A and B into C, i.e. $f(A, B) \to C$, which can also be expressed from a topological point of view by the homeomorphism condition of the following figures:

$$A \equiv C \backslash B \tag{2.3}$$

What physical phenomenon can the formula (2.1) correspond to? It models a certain system made up of subsystems that are not connected but share a common resource in a certain way. From the point of view of school knowledge, the laws of thermodynamics are of particular interest when looking for evidence for the ABC conjecture. A common resource can be *energy, phase space, entropy,* number of particles and other *integrals of motion.*

2.2 Microstate and macrostate of the system

Consider a system consisting of a mixture of ideal gases whose molecules do not interact with each other. In particular, this gives rise to *Dalton's Law*, which states that the pressure of a mixture of chemically non-reacting gases is equal to the sum of the partial pressures of each of the gases - and this is clear because of the absence of intermolecular interaction.

The author proposes to consider the ABC conjecture from the perspective of the *phase space* of a thermodynamic system with 2s dimensions, along the coordinate axes of which the values of s generalised coordinates q and s impulses p of this system (s is the number of degrees of freedom) are stored in the Hamiltonian, known to a student of mechanics under the name of the *kinetic energy theorem.* The operator form of the Hamiltonian in canonical form is quite beautiful, universal, described in the literature, but requires little experience in working with partial derivatives [13]. More precisely, the application of the statistical approach in physics requires knowledge of the basics of quantum mechanics [6-7], an understanding of the Heisenberg uncertainty principle $\Delta p \Delta q \geqslant \hbar/2$, an idea of the *Schrodinger equation*, the *statistical matrix*, the *Hamiltonian* operator, knowledge of the remarkable properties of *Hermitian* operators, the rules of operation with switches and *Poisson's brackets*, the basic formulas of thermodynamics, but the presentation below will be based mainly on knowledge of the material in the school curriculum.

Let's try to guess which class of physical phenomena corresponds to the ABC conjecture? Let's start with the concept of a microstate - this is a specific configuration of the system that describes the exact positions and impulses of all the individual particles that make up the system. Each microstate has a certain probability of occurrence as the molecules move. *The macrostate* of a system refers to its macroscopic properties such as temperature, pressure, volume and density.

2.3 Phase space. Probability distribution density

Each point in the phase space corresponds to a specific microstate of the system. During the evolution of the system, the phase point representing it describes a certain phase trajectory in the phase space. The product of differentials:

$$dqdp = dq_1 dq_2...dq_s dp_1 dp_2...dp_s \quad (2.4)$$

It is considered as a "volume element" of the phase space $\Delta\Gamma$. With regard to the formula under study, taking into account the shape of the cuboid and the equality of the increment of the linear function to its differential, the sign Δ will be used next along with the designation of the differential.

Let's express the probability and probability density ρ to detect an object in the given coordinates of the phase space using the obvious formula:

$$dw = \rho(q_1, q_2, ...q_s, p_1, p_2, \ldots p_s)dpdq \quad (2.5)$$

- the probability of finding the object under study in the specified infinitesimal intervals of coordinates and pulses within: $q_i \ldots q_i + dq_i, pi \ldots p_i + dp_i$. In this case, the volume of the phase space is expressed in terms of energy according to the formula:

$$\Delta\Gamma = \frac{d\Gamma(\bar{E})}{dE}\Delta E \quad (2.6)$$

Above, $\Delta\Gamma(E)$ denotes the number of quantum states with energies less than or equal

to E. Next, the probability density ρ is given and the normalization condition is written:

$$\sum_{\Omega} \rho(\bar{E})\Delta q \Delta p = 1 \quad (2.7)$$

the integral over all probabilities of the subsystem states forms *a complete group of events* $\Rightarrow$ for the probability density function W(E):

$$\int W(E)dE = 1 \cdots \sum W\Delta\Gamma = 1 \quad (2.8)$$

The formulae given above help to find the number of states of interest with energy between $E \ldots E + dE$. It is assumed that an inertial frame of reference is chosen in which the total momentum and angular momentum of the entire subsystem under consideration is zero. The dash above the energy $\bar{E}$ denotes the average energy value of the subsystem for all quantum states. Here and below the notation Δq and Δp are products of the form (2.4) over all degrees of freedom.

2.4 The quantum nature of prime numbers

Due to the nature of quantum mechanics and the statistical physics based on it, we can only talk about finding the probability density distribution for coordinates or momentum separately, and not for both together, since coordinates and momentum of a particle in general cannot have certain values at the same time due to the Heisenberg uncertainty principle. The desired probability distributions should take into account both the statistical uncertainty and the uncertainty directly inherent in the quantum mechanical description of $\Rightarrow$. The answer to the question whether a point belongs to an arbitrary manifold in the studied construction of cuboids $1^n \in A$ or $1^n \notin A$ in phase space should be sought from the point of view of statistical physics / probability theory.

2.5 Preservation of phase volume. Entropy

According to *Liouville's theorem*, the volume of phase space has the property of invariance with respect to canonical transformations preserving *Hamiltonian* and *integrals of motion*, in particular energy, momentum, angular momentum. [5, pp. 192-193].

For the quasi-classical case, where the laws of quantum mechanics can be replaced by Hamilton's equations, which are a reformulation of the well-known Newtonian equations. At the same time, the volume of the phase space

$$\Delta \Gamma = \frac{\Delta q \Delta p}{(2\pi)^s} \qquad (2.9)$$

The value of $\Delta \Gamma$ is called the *statistical weight* of the microscopic state of the subsystem, and its logarithm is called the entropy. The entropy, in turn, is determined by the mean value, denoted by brackets $<>$, of the logarithm $\Delta \Gamma$, both in the case of the description of the system under study from the position of its quantum state, from the probability density distribution function in energy E_i, and in the quasi-classical approach, from the probability density, according to the formulae

$$S = - \langle lgw(E_i) \langle = - \sum_i w_i ln(w_i) \quad (2.10)$$

$$S = - \langle ln\left[(2\pi\hbar)^s \rho\right] \langle = $$
$$- \int \rho \cdot ln\left[(2\pi\hbar)^s \rho\right] dpdq \quad (2.11)$$

The entropy defined in this way is, like the statistical weight itself, a dimensionless quantity. (In school physics and in some textbooks of theoretical physics, the *Boltzmann constant* k_B is placed in front of the logarithmic sign in the last formula $= 1.38 * 10^{23}$ J/K, which allows you to measure the absolute temperature in Kelvins rather than in energy units, but with regard to the hypothesis, the Boltzmann constant will be more convenient

in some cases to omit cases in which is easy to guess from the context).

For simplicity, consider a closed system as a whole (i.e. isolated from other systems, e.g. placed in a thermostat), where $\Delta \Gamma_1$, $\Delta \Gamma_2$, $\Delta \Gamma_3$ are defined.... - statistical weights of its different subsystems. If each of the subsystems can be in one of the $\Delta \Gamma_\alpha$ states, then the phase volumes of the subsystems are multiplied and the entropies of the subsystems add up, which is illustrated by the expressions:

$$\Delta \Gamma = \prod_\alpha \Delta \Gamma_\alpha; \ldots S = \sum_\alpha S_\alpha \qquad (2.12)$$

In other words, the formula describing the phase space allows the separation of the variables. In practice, it is often necessary to deal with cases where not all of the microscopic particle motion is quasi-classical, but only the motion corresponding to some of the degrees of freedom, while for the rest of the degrees of freedom the motion is quantum (for example, the translational motion of molecules can be quasi-classical with the quantum nature of the intramolecular motion of atoms). The formula 2.1 can be written as

$$e^{\alpha_1 x_1} e^{\alpha_2 x_2} \ldots e^{\alpha_k x_k} + e^{\beta_1 y_1} e^{\beta_2 y_2} \ldots e^{\beta_k y_l}$$
$$= e^{\gamma_1 z_1} e^{\gamma_2 z_2} \ldots e^{\gamma_k z_l} \quad (2.13)$$

where the exponents of x, y, z for each indexes are chosen so that only one prime number is chosen from those listed in the formula (2.1), which is easily achieved by taking the natural logarithm. Looking ahead, we note that in classical mechanics the variable w_k on which entropy depends cannot be chosen unambiguously, since in classical physics entropy is determined up to an arbitrary term.

The choice of the generalised momentum Δp_i or the generalised coordinate Δq_i for the role of the multiplier considered in the formula (2.5) is arbitrary. For the sake of simplicity, we limit the volume of the studied

gas mixture to q units and consider all Δq_i = 1, i.e. we choose an element of small volume in the system that is numerically equal to one (nanometres, angstroms or an atomic unit of length a_0 used in atomic and quantum physics, the so-called Borovsky radius - the closest to the nucleus of the orbit of the unexcited electron of the hydrogen atom H in the atomic model $5.29 \cdot 10^{-11}$ m).

The ABC conjecture can be considered as a mathematical interpretation of a complex system consisting of simple subsystems, where the principle of additivity of entropy and multiplication of volumes of phase spaces of subsystems is fulfilled. The formula (2.12) is suitable for expressing the law of conservation of entropy of a subsystem in the separation of variables and adiabatic processes, which are quite slow compared to the relaxation time / establishment of local equilibrium. In practice, under laboratory conditions, this time passes quickly and is comparable to the time of propagation of a sound wave in a vessel filled with gas.

Speaking of the exponent in the formula (2.13), we note that the well-known formulas for the distribution of the probability density of finding particles are expressed in terms of an exponent: as in the *Boltzmann distribution* studied in physics and mathematics schools $n_0 e^{-\frac{mgh}{k_B T}}$ - the so-called *barometric formula* - and *Maxwell's distribution*:

$$dw = \left(\frac{m}{2\pi k_B T} \right)^{3/2} e^{-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2k_B T}} dv_x dv_y dv_z$$
$$(2.14)$$

Both formulae give an idea of the *statistical ensemble* that should be considered in more detail.

2.6 The Canonical Gibbs Distribution

The *statistical ensemble* is represented in phase space by a set of points whose distribution is described by the probability density. It gives an answer about the probability of

finding a state of the whole system in which a given body in a certain quantum state w_n with energy E_n is in a state described in a microscopic way. The microscopic state of the external environment is not observed. *The Canonical Gibbs distribution* is based on a simple principle: the probability of finding a complex system 1-2 is equal to the product of the probabilities of its subsystems 1 and 2: $\rho_{12} = \rho_1 * \rho_2$ (which implies the convenience of using logarithms to study a complex system). Based on this postulate of equal probability of all microstates available to the system under study, which follows from *The Symmetry Principle*, Gibbs derived the formula:

$$w_n = exp \left(\frac{F - E_n}{T} \right), \cdots$$
$$\rho = (2\pi h)^s exp \frac{F - E(p,q)}{T} \quad (2.15)$$

Before the ellipsis above, the probability distribution function for the energy of the system in the state E_n is given in the quantum mechanical representation, after that in the quasi-classical representation.

Where

- $w_n(p,q)$ - the probability of being in a state corresponding to a wave function with eigenvalue
- E_n. - The energy as a function of the generalised coordinates,
- s is the number of degrees of freedom.
- F is the free energy of the gas. (In thermodynamics, work is done by the difference between the free energies of the final and initial states of the gas, which explains the convenience of using this variable).
- ρ - Probability density of being in a given phase difference,

It is interesting to note that the free energy of a gas is a simple consequence of the *normalisation condition* of the above canonical Gibbs distribution. This is a clear example

of the connection between mathematics, theoretical physics and laboratory experiments. In thermodynamics, free energy F, entropy S, energy E, volume V, pressure P, temperature T and a number of other variables are derived from and controlled by each other. The first three variables listed are <u>additive</u> functions.

It is possible that not all the motion of a microscopic particle is quasi-classical, but only the motion corresponding to some of the degrees of freedom, while for the rest of the degrees of freedom the motion is quantum. For example, the translational motion of molecules can be quasi-classical with the quantum nature of the intramolecular motion of atoms. In this case, the energy levels of the body can be written as functions of the quasi-classical coordinates and momentum $E(p, q)$.

The Boltzmann and Maxwell distributions mentioned above are special cases of the canonical Gibbs distribution. In the first case the potential energy is replaced by a negative sign in the numerator of the exponent fraction, in the second case by the kinetic energy. In both cases the denominator remains T or kT.

Due to the identity of the gas molecules, when analysing the Maxwell distribution, it is sufficient to consider the phase space of only one molecule and to express the energy in the impulse representation in coordinates $E = \frac{1}{2}m(px^2 + py^2 + pz^2) + \epsilon_k$, where ϵ_k is the energy of a gas divided by the number of gas molecules, the k-th energy level of a molecule due to its rotational and vibrational degrees of freedom, the intrinsic angular momentum of elementary particles, *spin*, etc. Furthermore, integration in the generalised coordinates of the dq system can be replaced by simple multiplication by the volume of the vessel - a similar algorithm is used for the Maxwell distribution (2.14). This greatly simplifies the calculations. As a result, the following formula is used to find the free energy F of an ideal gas (remember that the difference *free energy* of the final and initial

states is just the perfect work due to the gas) [8 forms 31.3, 31.2]:

$$F = -NTln\left[\frac{eV}{N}\left(\frac{mT}{2\pi\hbar^2}\right)^{3/2} Z\right]$$
$$= -NTln\frac{eV}{N} + Nf(T) \quad (2.16)$$

Where

- N - number of molecules
- T is the absolute temperature in energy units
- e is the Euler's number 2.718 ...
- V is the volume of the vessel, where m is the mass of the molecule
- $\hbar$ - the reduced Planck constant equal to $1.055 * 10^{-34}$ J*sec.
- Z is a statistical sum depending on the energy levels of the molecule, its definition will be given below.

The initial temperature, volume and number of particles are considered as external parameters of the system under study. An alternative representation of the above formula is to put all the parameters into a separate function that depends only on the temperature $f(T)$.

The entropy of S is determined from thermodynamic relations as a partial derivative with a minus sign of the free energy:

$$S = -\frac{\partial F}{\partial T} = Nln\frac{eV}{N} - N\frac{df}{dT}(T) \quad (2.17)$$

On the basis of the fundamental equations of thermodynamics, it is possible to find other thermodynamic variables of the subsystem under study, rather than to completely determine or set its macroscopic state.

2.7 Degeneration of energy levels

When different wave functions have the same eigenvalue of the energy E_n, it means that the energy level is degenerate. The exponent in the formula 2.1 corresponds to the multiplicity of the energy level degeneracy, called

the statistical weight g_k. In this case, part of the formula (2.15) contains a statistical sum determined by the formula [9, p. 35 of form 1.71b].

$$Z = \sum_k g_k e^{-\epsilon_k/kT} \qquad (2.18)$$

The normalisation condition for the formula (2.16) allows us to calculate the free energy F from the probabilities of the quantum states:

$$1 = \sum_n^N w_n = e^{F/T} \sum_n^N e^{-E_n/T}$$

$$\rightarrow F = -Tln \sum_n^N e^{-E_n/T} \quad (2.19)$$

Let's use the general formula (2.16) to calculate the free energy of an ideal gas obeying Boltzmann's statistics. By writing the energy E_p as the sum of the energies of the molecules ϵ_k, it is possible to reduce the summation over all states of the gas to the summation over all states of a single molecule, as in the case of the Maxwell distribution. Each state of the gas is determined by a set of N (N is the number of molecules in the gas) values of ϵ_k, which in the Boltzmann case can all be considered to be different from each other (in each molecular state - no more than one molecule, due to the limitations imposed by the wave functions for fermions, since it is with such objects that the barometric formula works). If we write $e^{-\frac{E_n}{T}}$ as a product of the multipliers of $e^{-\frac{\epsilon_k}{T}}$ for $\forall$ of molecules and sum independently over all states of each molecule, we get an expression corresponding to N

$$\left(\sum_k e^{-\epsilon_k/T} \right)^N = \sum_n e^{-E_n/T} =$$

$$\frac{1}{N!} \left(\sum_k e^{-\epsilon_k/T} \right)^N \quad (2.20)$$

At the same time, all sets of N different values, differing only in the distribution of

identical gas molecules over the levels of ϵ_k, correspond to the same quantum state of the gas. However, in the statistical sum, in the formula above, each of the states should be taken into account only once. Therefore, we must also divide the expression obtained by the number of possible permutations of N molecules with each other, i.e. by $N!$, and evaluate the value with *Stirling formula* $n! = \sqrt{2\pi n} \left(\frac{n}{e} \right)^n$.

As a result of this estimation, we obtain a formula for the free energy of a Boltzmann ideal gas:

$$F = -NTln \left[\frac{e}{N} \sum_k e^{-\epsilon_k/T} \right] \qquad (2.21)$$

In the case of degenerate energy levels, the number of repetitions of the corresponding value is ϵ_k, it is equal to the multiplicity of the degeneracy. What happens to the formulae (2.17) and (2.18) for a mixture of gases? To answer this question we turn to the *Large canonical Gibbs distribution* with a variable number of particles:

$$w_{n,N} = A \dot{e} xp \frac{\Omega + \mu N - E_{n,N}}{T} \qquad (2.22)$$

Here above is the distribution function of the subsystem of an ideal gas over two variables:

- for the energy value E_n and the number of particles N
- Ω - thermodynamic potential
- T - absolute temperature in energy units
- μ is the chemical potential of the molecule.

Now let's define the distribution function of the subsystem of a mixture of ideal gases of a gas by variables - the energy value E_n and the number of particles N_1, N_2, N_3 - degenerate energy levels are also possible.

$$w_{n,N_1,N2...} = A exp \frac{\Omega + \sum_i \mu_i N_i - E_{n,N_1,N2...}}{T}$$
$$(2.23)$$

The energy of E_p can also be represented as the sum of the energies of the ϵ_k molecules,

reducing the summation over all gas states to the summation over all states of a single molecule.

For a mixture of ideal gases, the above formulae remain valid, only the number of molecules of the corresponding gas is replaced by N [9]. For a mixture of gases, the following thermodynamic variables have additive properties: number of molecules, energy, entropy, partial pressure of each gas, thermodynamic potential, and a number of others. In this case, according to the rules of thermodynamics, the volume and temperature are the same for all the molecules of each gas in the mixture. Note that, due to the absence of interaction between molecules, the entropy values are summed up and, accordingly, the volumes of the phase spaces are multiplied (2.12).

It is easy to trace the analogy of the formula (2.23) with the logarithm of the formula (2.1) and check their similarity. Although the formulas have slight differences, this should not be discouraging. This confidence is supported by a formula modelling phase space and entropy (2.10). Pairs of products in (2.23) $\mu_i N_i$ under the sign of the sum enter symmetrically. In this formula it is possible to sum all states of energy levels, not molecules. The state of the energy levels is determined by the quantum-atomic parameters of the molecule, which you learned in high school chemistry.

2.8 The role of radicals of the numbers A, B, C

The ABC conjecture works with the concept of *the quality of a triple of numbers* based on the decimal logarithm. (Let us assume a transition to the natural logarithm by simultaneously dividing the numerator and the denominator by ln10). For example, the radical of the product of the numbers $Rad(ABC)$ corresponds to a mixture of ideal gases with non-degenerate energy levels, which is achieved at a sufficiently low temperature when only the translational and rotational degrees of

freedom for polyatomic molecules are activated. Considering the above, this means the first power of the prime numbers of the formula (2.1).

Due to the *Clapeyron equation* $PV = kNT$ it is possible to control the temperature, the gas pressure by adjusting the initial temperature, the number of molecules in the mixture (concentration) and the volume of the subsystem. Thermodynamic relations are used to easily calculate the energy of the gas molecules and the entropy, and hence the volume of the phase space $\Delta\Gamma_i = exp(S(E_i))$. Next we can calculate $\Delta\Gamma_1, \Delta\Gamma_2, \Delta\Gamma_3 \ldots$ - The statistical weights of the gases 1, 2, 3 in the mixture are each equal to the prime numbers of the formula (2.1) of the first degree - the radical $Rad(ABC)$. Let's denote such a value of the volume of the phase space under $\Delta\Gamma_0$ and the corresponding entropy $S_0 = \frac{lnRad(ABC)}{L}$, in the denominator of this fraction it is necessary, according to the definition of entropy (2.10), and the length of this diagonal or the total number of elements L = k+l+m is calculated from the formula 2.1 and is equal to the total number of prime factors in the ABC conjecture. If each of the subsystems can be in one of the Γ_α quantum states, then the phase volumes of the subsystems are multiplied and the entropies of the subsystems add up.

In the case of adiabatic compression, the basic thermodynamic equation takes the form that the internal energy of the gas decreases (increases) only by the amount of work done by the gas (over the gas). According to the formula of the first law of thermodynamics $dQ = dE + PdV$, where dQ is the amount of heat, P is the gas pressure, dV is a small increase in volume, the multiplier of PdV is equal to the work done. In an adiabatic process $dQ = 0$ there is no heat exchange, there are no *dissipative* processes, and therefore all changes in the subsystem remain reversible and $dE = -PdV$.

It is known from school thermodynamics that, in an adiabatic process, the relationship between pressure and volume / temperature

and volume has the form

$$PV^\gamma = const \dots TV^{\gamma-1} = const \quad (2.24)$$

where $\gamma > 1$ is the quotient of the heat capacity at constant pressure divided by the heat capacity at constant gas volume $\frac{C_P}{C_V}$. In adiabatic compression the entropy of the system remains constant, but degenerate energy levels ϵ_k are "switched on" due to an increase in temperature.

This is accompanied by the appearance of degrees above one in the formula (2.1). Let's denote such a statistical state of the system by the letter $\tilde{D}$, which symbolises the appearance of degenerate energy levels or entropy, $S_{\tilde{D}}$. (Details are given below). Next, we divide the phase volume corresponding to some generalised coordinates or impulses (e.g. those associated with quantum effects) into separate subsets of the studied phase space: A, B and C, according to the indices given in the formula (2.1). As mentioned above, entropy is an additive function). Due to the adiabatic compression, which preserves the entropy, and the conditions of the experiment, we obtain

$$q_{ABC_{max}} =$$
$$\frac{ln(C)}{lnRad(A) + lnRad(C-A) + lnRad(C)}$$
$$= \frac{S_C \cdot m}{S_0 L} = \frac{S_C \cdot m}{(S_A + S_B + S_C)L} < 1 \quad (2.25)$$

In this formula, an upper bound is placed on q_{ABC} by formulating the ABC conjecture. After the second equal sign there is a transition to thermodynamic (statistical) variables. The denominator of the numerator contains the value of the entropy of the phase space, taking into account the multipliers m and $L = (k + l + m)$, expressed by the definition of the entropy of the subsystem C as part of the studied system D under the conditions of an adiabatic process.

The problem of finding the extremum of a function in the form of the next first order

fraction is reduced to finding the maximum and minimum values of the denominator, where the variable part is the number A, while $C = Const$. => the minimum q_{ABC} is obtained at $A = C/2$, which is easy to understand from the symmetric occurrence of the variable part of the expression under the sign of the logarithm in the denominator, analysing a function of the form $y = x(c - x)$.

Remember that (A is to some extent a product of prime numbers). The task is to get the correct evaluation. The maximum term in the denominator of the ABC conjecture and the formula will be $q_{ABC_{max}} < 1$. The entropy does not change during the adiabatic process.

Meanwhile, numerical experiments have found a number of maximum values of q_{ABC} above one. A volunteer network called ABC@home is dedicated to a distributed computing project that was supposed to compile a complete list of all ABC triples consisting of integers consisting of a maximum of 18 digits. In 2011, the project achieved this goal by compiling a list of 14,482,065 triples. See There are 241 ABC triples of "quality" q not lower than 1.4, which are often called "good" ABC triples [11,14].

How can this paradox be explained? - The author is convinced that this phenomenon is caused by *fluctuations*, as explained below in a section specifically dedicated to this phenomenon.

3 Linear Spaces and Polynomials Functions

To find a proof of the ABC conjecture by mathematical methods, it is necessary to use the rules of linear algebra, representations of quantum mechanics about the wave function, the operator form of the Hamiltonian [7, 1.18], estimates of entropy using a statistical matrix [8, pp. 28-29] in a quantum mechanical representation. The rapid development of quantum computing and cryptography is

helping to popularise knowledge of quantum mechanics. In order to broaden the horizons of a student of the XXI century, it would be logical to add the basics of quantum mechanics to the secondary school curriculum, using an analogy with radio waves, signal spectra. For many years, the electronic configuration of an atom, the arrangement of electrons in an atom according to levels and sublevels, has been studied in school chemistry classes on the basis of general concepts of quantum mechanics. Based on the conditions of the ABC conjecture, an additive function should be chosen. The most suitable candidate is entropy, defined as the logarithm of the phase space (2.10). For adiabatic processes the volume of the phase space is preserved. This simple statement is the key to the proof and requires almost no mathematical calculations.

The formula (2.1) implies a statistical description of a physical system consisting of subsystems expressing the additive property of a common resource for energy, entropy and other thermodynamic quantities. These properties are provided by the decomposition of integers *A, B, C* into prime numbers for which *Greatest Common Divisor* (GCD) = 1. It is known from the introductory course in number theory that the rules apply to the ring of integer polynomials in a given field of complex numbers k[x]. Such polynomials have a number of properties similar to those of integers. There is a homomorphism between the algebra of integers and the algebra of polynomials for addition and multiplication operations. In the following, the terms simple and irreducible are used synonymously when referring to polynomials.

Let $f \rightarrow k[x]$, then there is a one-to-one decomposition:

$$f = c \prod_p p^{\alpha(p)}, \qquad (3.1)$$

where c is a constant and the product is taken by irreducible polynomials of the form $(\lambda - a_i)^k$ [10 Chapter 1 Theorem 2]. The degrees and the constant are uniquely determined.

$$(\lambda - a_1)^{\alpha_1}(\lambda - a_2)^{\alpha_2} \cdots (\lambda - a_k)^{\alpha_k} q_1(\lambda)$$
$$+ (\lambda - b_1)^{\beta_1}(\lambda - b_2)^{\beta_2} \cdots (\lambda - b_l)^{\beta_l} q_2(\lambda)$$
$$+ (\lambda - c_1)^{\gamma_1}(\lambda - c_2)^{\gamma_2} \cdots (\lambda - c_m)^{\gamma_m} q_3(\lambda) = 1$$
$$(3.2)$$

Integer polynomials are written in the formula above, based on irreducible ones, they are also mutually simple [12, p. 333], which is briefly expressed in the formula above: there are polynomials q_1, q_2, q_3, the total sum of the formula (3.1) assuming GCD = 1.

Each of the above polynomials can be uniquely mapped onto a linear map $\mathcal{A}$ (respectively $\mathcal{B}$ and $\mathcal{C}$) with a characteristic polynomial [12,13] whose roots as *eigenvalues* are taken from formula (3.1) - these will be just the primes contained in $Rad(ABC)$. If a linear transformation of dimension n has n independent *eigenvectors*, then using these vectors as a basis, we will reduce the square matrix of the linear transformation to a diagonal form, while the eigenvectors corresponding to different eigenvalues will be linearly independent, and we can choose a basis in which such vectors will be orthogonal. In the case of multiple roots we have:

$$(\lambda \mathbf{I} - \mathcal{A})^k \mathbf{x} = 0 \qquad (3.3)$$

By elementary transformations, the linear mapping matrix can be reduced to a Jordan form of size $k \star k$ corresponding to the multiplicity k (height) of the *root* [12,13]. Putting it all together, it is possible to obtain a linear mapping whose characteristic polynomial is the product of the characteristic polynomials of the linear mappings from formula (3.1). The matrix corresponding to this combined linear mapping will have the block-diagonal form $\left(\begin{smallmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{smallmatrix} \right)$. The eigenvalues of a matrix are the result of combining the eigenvalues of the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$. The result is a unique *decomposition* of the space R into a *direct sum of invariant subspaces*:

$$R = R_1 \bigoplus R_2 \bigoplus R_3 \bigoplus R_4 \bigoplus R_5 \cdots \bigoplus R_i$$

$$(3.4)$$

The set of all root vectors corresponding to the eigenvalue of λ forms an invariant subspace of R_i. This linear space is described by the matrix $\widetilde{D}$.

We set the determinant of this matrix and its characteristic polynomial to zero in order to search for eigenvalues and vectors:

$$Det\widetilde{D} = \prod_{i=1}^{i=L} (\lambda - \lambda_i)^{k_i} = 0 \qquad (3.5)$$

where L is the number of primes in the formula (2.1) λ_i is the proper/root value of the linear mapping $\widetilde{D}$ and hence of the matrices $\widetilde{D}$. It is known from the algebraic theory of matrices that elementary transformations can be performed on λ matrices to produce similar matrices, but at the same time the trace of the matrix remains constant, i.e. the sum of its elements on the main diagonal and the determinant det - both quantities do not depend on the basis and are invariant.

Denote by $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots \lambda_i$ the eigenvalues of the matrix, they are just taken from a polynomial (3.2) with roots from a set of prime numbers (2.1), but using the logarithm function and averaging over the main diagonal of the matrix (for more information, see the formula (3.7)), all these eigenvalues are <u>different,</u> non-zero under the conditions of the ABC conjecture. It is known that the determinant of the matrix is equal to the product of the eigenvalues. In our case, from the point of view of thermodynamics, $det\widetilde{D}$ corresponds to the phase space of the studied system with unexcited energy levels $\Gamma_0 = Rad(ABC) = det\widetilde{D}$.

Since we are considering square matrices, and due to the *matrix commutability* property of ourselves, in a number of cases, including the one under study, it is possible to raise the λ matrix $\widetilde{D}$ to a power, perform algebraic operations on it, calculate polynomials from the matrix.

This illustrates the general rule of *homomorphism* (A homomorphism in the category of algebraic systems is a mapping of an algebraic system A to B that preserves the basic operations and basic relations).

$$A =< A, \phi > B =< B, \psi >, f : A \to B).$$

In general, scalar functions can be computed from a square matrix by decomposition into a Taylor series, but with some peculiarities [13, 182-183]. In the theory of analytic functions (i.e. having a derivative in the complex plane, which leads to an amazing conclusion about the infinite differentiability of an analytic function [13]). It is known that a series of a complex variable has the property of converging on a certain convergence ring $r < |z| < R$ on the complex plane $\mathbb{C}$ [13, p. 64], with the exception of *poles*.

3.1 Logarithm of the Matrix

Let's calculate the logarithm of the matrix $\widetilde{D}$ with root vectors corresponding to (3.2). Here, different eigenvectors/root vectors are chosen as the basis, which are prime numbers, and therefore the matrix will have a *block-diagonal form* (the order of the Jordan cells is not essential). The F function works with each *Jordan cell* separately[15 p. 182]. From this decomposition it is easy to conclude that a cellular/blockwise comparison of functions from matrices is possible, which will be used below.

It is quite easy to calculate the logarithm [13,15] from a Jordan cell by applying the decomposition in one row <u>Newton-Mercator</u> (Taylor series for the function of the natural logarithm of the matrix) and then represent the matrix to the right of the equal sign as the sum of the unit matrix $\mathbf{I}$ and the matrix $\mathbf{G}$ with zero values on the diagonals: $\mathbf{G} = \lambda(\mathbf{I} + \mathcal{D})$. Applying the well-known Taylor series expansion of the logarithm of a scalar variable and taking into account the above, which also applies to the matrix $\mathbf{G}$, we have

$$ln(I + \mathbf{G}) = \frac{\mathbf{G}}{1} - \frac{\mathbf{G^2}}{2!} + \frac{\mathbf{G^3}}{3!} - \frac{\mathbf{G^4}}{4!} \cdots \; (3.6)$$

This series ends at some degree, since the Jordan cell is *nilpotent* and, when raised to the power of k (the dimension of the Jordan cell), becomes a zero matrix.

As a result, it is easy to calculate the entropy of $S = ln\Delta\Gamma$ by averaging over all microcanonical states. It is important to take into account the normalisation condition - the sum of all probabilities of the states of the system under study forms <u>*a complete group of events*</u> $=> \sum_{i=1}^{i=L} \lambda_i = 1$. Here the probability is expressed from the eigenvalue λ_i, which is uniquely associated with the energy of the microstate, summation is performed over all possible microstates, and non-zero values are on the main diagonal of the matrix $\widetilde{\mathcal{D}}$. The length of this diagonal is $L = k + l + m$ - the number of simple factors in the ABC conjecture, see formula (2.1).

Initial entropy value $S_0 = \dfrac{lnRad(ABC)}{L}$ The logarithm of the product is decomposed into the sum of logarithms and is equal to the mean value from the trace of the entropy matrix $=>$ divided by the number of non-zero values on the main diagonal of the matrix $\mathcal{D}$, i.e. the length of the chain L in the formula (2.1). Let us switch on the degeneration of the energy levels at time t_1, which in the language of matrix algebra means the appearance of multiple eigenvalues - root vectors (3.3). This can be realised by "including" powers greater than one in the characteristic polynomial:

$$\frac{1}{L}\prod_{i=1}^{i=L} \frac{e^{-\frac{E_i}{T}} \cdot (\lambda - \lambda_i)^{1+\sigma(t-t_1)(k_i-1)}}{1+\sigma(t-t_1)(k_i-1)} \quad (3.7)$$

and then the matrix corresponding to this characteristic polynomial is brought into Jordan form and the logarithm is taken from the resulting matrix. Let's explain the meaning of the variables:

- λ_i - all prime numbers are multipliers of the L-series of numbers Rad(ABC), they are also eigenvalues of the linear mapping /matrix

- k_i is the exponent of the formula (3.2), at the same time the multiplicity of the root and the size of the i-th Jordan cell of the linear map.
- σ - function <u>*sigmoid*</u> $\sigma(t) = \frac{1}{1+e^{-\nu(t-t_1)}}$ can be approximated by the formula where the parameter ν is set from the outside and regulates the smoothness of the change, the higher the value of ν, the steeper the "step" at time t_1. This function is defined here on the real axis $\mathbb{R}$.
- E_i is the energy calculated by the microstate i. This factor, divided exponentially by the temperature T, determines the density of the probability distribution depending on the Hamiltonian of the system under study.

Above, according to the formula (3.4), it was concluded that it is possible to decompose the linear space into a direct sum of subspaces, i.e. to reduce the matrix to Jordan form. For each cell there is a unique root λ_i. Since the multiplier $e^{-\frac{E_i}{T}}$ remains the same within each Jordan cell, it can be considered as a constant and reduced in the following arguments ("cell-by-cell" / "block-by-block" comparison of matrix functions).

From the point of view of school physics, it can be seen that when studying an ideal Carnot machine, the calculation of its efficiency is based on the value $\frac{\delta Q}{T}$, which is the entropy differential. Consequently, the formula (3.7), taking into account the definition of entropy (2.10), operates with probabilistic quantities, which in turn is a purely mathematical concept that excludes such an unusual quantity for the algebra of matrices as temperature (the latter can be determined for the system under study from the canonical definition of Gibbs by taking a quotient, the derivative of entropy with respect to energy).

The function (3.7) is differentiable and can be analytically extended to the complex plane. This means that it can be positioned along the main diagonal of the Jordan form

and the matrix logarithm calculated from the result. This function corresponds to the volume of the phase space. Reducing the matrix just mentioned to the Jordan form, taking the logarithm of each element on the main diagonal of the matrix and averaging over all the values will give the entropy value of the system under consideration.

In fact, a careful look at (3.7) reveals the formula (2.10) for finding the entropy. The average over all microstates is obtained by dividing by the total number of microstates. Each microstate is considered equally likely. When averaging the logarithm of the matrix, the sign of the product in the formula (3.7) is replaced by the sum. The somewhat cumbersome denominator of the fraction after the product sign in this formula allows you to first average the value of the logarithm of the element on the main diagonal separately for each Jordan cell, taking into account the multiplicity (degree) of each multiplier. According to the results of such cellular averaging, the exponent of each prime number from the matrix is reduced. Next, averaging is performed over the entire diagonal of the matrix, where the unique value of each prime number occurs only once, and there are L such numbers in total. There is a "smearing" of the probability density from the averaging operation over all possible Jordan cells and taking into account the multiplicity of the root vector λ_k, i.e. its height or exponent of a prime number in the ABC conjecture formula.

$$S = - \langle lnw(\lambda_i) \rangle = - \sum_i w_i ln(w_i)$$

$$= \frac{1}{L} \sum_{i=1}^{i=L} \frac{1}{k} ln(\lambda_i^k) e^{-\frac{E(\lambda_k)}{T}} \quad (3.8)$$

With adiabatic compression/expansion, the volume of the phase space of the system and the averaged matrix logarithm of it - the entropy - remain constant. The multiplier $e^{-\frac{\epsilon_k}{T}}$ does not qualitatively change the main conclusion: the conservation of entropy as a result of an adiabatic change in the macrostate of the system. In this case the trace of the matrix $\widetilde{\mathcal{D}}$ is equal to the sum of the traces of the combined matrices. It is possible to compare the initial value of the entropy S_0 with the value at time $t_1 + 0$ and then select some degrees of freedom of the system under study separately. The situation was considered above when not all the motion of the system is quasi-classical, but only the motion corresponding to a part of the degrees of freedom, while for the rest of the degrees of freedom the motion is quantum. Since entropy is preserved during the adiabatic process, it is possible to return the subsystem to its original state.

The cumbersome formula (3.7) and the formula (3.8) based on knowledge of statistical physics can be replaced by the adiabatic equation from the school course (2.24), which is limited to a general description of degenerate energy levels and, accordingly, the appearance of Jordan cells greater than one, which greatly simplifies understanding the essence of the proof.

3.2 Trace of matrices. Entropy

Thus, a simple matrix relation can be formulated in the form of the following equations

$$Tr(\mathbf{D}) = Tr(\mathbf{A}) + Tr(\mathbf{B}) + Tr(\mathbf{C}) \quad (3.9)$$

which is obvious from the analysis of the main diagonal of the matrix $\widetilde{\mathcal{D}}$ of the block diagonal form $\left(\begin{smallmatrix} \mathbf{A} & 0 & 0 \\ 0 & \mathbf{B} & 0 \\ 0 & 0 & \mathbf{C} \end{smallmatrix} \right)$.

And the following formula means averaging statistical variables taking into account the weights of the terms (diagonal lengths $L = k + l + m$ of each block of the matrix $\widetilde{\mathcal{D}}$.

$$\frac{1}{L} Tr(\tilde{D}) = \frac{k}{L} \frac{Tr(\mathbf{A})}{k} + \frac{l}{L} \frac{Tr(\mathbf{B})}{l} + \frac{m}{L} \frac{Tr(\mathbf{C})}{m}$$
$$(3.10)$$

The latter implies the equality $S_D = S_A + S_B + S_C$ or in terms of matrix traces:

$$Tr(\hat{w}(ln\hat{w}))_{\tilde{D}} = Tr(\hat{w}(ln\hat{w}))_A +$$
$$Tr(\hat{w}(ln\hat{w}))_B + Tr(\hat{w}(ln\hat{w}))_C \quad (3.11)$$

This simply expresses the properties of additivity of entropy, especially when variables are separated. According to the ABC conjecture $A + B = C$. From these relations, taking into account the additivity property of entropy and obeying the normal law of fluctuation of this value, it is easy to calculate q_{ABC}.

3.3 Statistical matrix for arbitrary A, B and C

It is possible to map a quantum mechanical system onto linear maps $\mathcal{A}, \mathcal{B}, \mathcal{C}$. Let's move from Euclidean space to Hilbert space.

Here, orthogonal vectors are an orthonormal system of functions, so that any operator $\hat{Q}$ is expressed in terms of $\langle n|\hat{Q}|m\rangle = \int \psi_n^* \hat{Q} \psi_m dq$ is explicitly time-independent and commutes with the Hamiltonian, then its matrix elements $\langle n|\hat{Q}|m\rangle$ do not change with time. This is the quantum form of motion integrals. It is important to note that in our case all the eigenvalues of λ, i.e. the energies of the microstates, are real, which means that the H *Hermite operator*. The remarkable property of the Hermite operator ensures the commutativity of the Hamiltonian with a number of operators, such as energy and entropy. Note that the time derivative of the statistical matrix commutes with the Hamiltonian:

$$\left[\hat{H}, \hat{w}\right] = 0 \quad (3.12)$$

For the physical phenomena under consideration, this condition is fulfilled. And the result is a quantum mechanical analogue of Liouville's theorem: the commutativity of an operator on any quantity with a Hamiltonian is precisely the quantum mechanical expression of the persistence of that quantity in time.

The eigenvalue of λ_i corresponds to the energy of the subsystem E_i. And as you can easily guess, the sum of the eigenvalues on the main diagonal is the energy of the whole system E (energy is also an additive variable)

The eigenvalue of λ_i corresponds to the energy of the subsystem E_i. And as you can easily guess, the sum of the eigenvalues on the main diagonal is the energy of the whole system E (energy is also an additive variable)

$$\hat{H}\psi = E\psi \quad (3.13)$$

The eigenvalues of the Hamiltonian of a system consisting of two subsystems are usually written as follows

$$\hat{H}|n, m\rangle = (E_1 + E_2)n.m > \quad (3.14)$$

This formula can be extended to the case of a number of subsystems and the analogy with invariant subspaces can be verified, see (3.4).

To find a statistical matrix for arbitrary numbers A, B, C, first assume that the system is in a pure quantum state with a wave function $\langle n|\hat{\psi}|m\langle$. The probability distribution for the coordinates is given by the square of the module:

$$|\psi|^2 = \sum c_m^* \cdot c_n \psi_m^* \psi_n \quad (3.15)$$

Where the product of the coefficients of the expansion of the wave function ψ according to the system of orthonormal (basis) c_m and c_n can be replaced by the probability w_{nm}. As a result, we get the following formula for the probability distribution by coordinates (keeping the diagonal elements of the statistical matrix)

$$\sum\sum w_{mn}\psi_m^* \cdot \psi_n = \sum \psi_n^* \hat{w} \psi_n$$

$$=> dw_q = \sum \psi_m^* \hat{w} \psi_m dq \quad (3.16)$$

In an expression written in this form, any complete system of normalised wave functions can be used as functions of ψ_n. The

task of determining the statistical distribution is reduced to calculating the probabilities $w_n = w_{nn}$, which represent the desired distribution function in quantum statistics. The formula for the mean of an arbitrary value f is simplified:

$$\langle f \rangle = \sum w_n f_{nn} \qquad (3.17)$$

Therefore, to find the entropy, it is sufficient to calculate the average of the trace from the statistical matrix (2.10).

3.4 Analogies from ODE systems

It is interesting to draw some analogies between the quantum mechanical approach described above and systems of ordinary differential equations.

Consider the fundamental matrix of a system of linear differential equations Y and the Vronsky determinant W = det|Y(t)|. According to the *Liouville-Ostrogradsky theorem* [13, section VIII(18)]:

$$W(t) = W(t_0) exp \int_{t_0}^{t} Tr D(\tau) d\tau$$

$$\cdots \prod_{1}^{n} \lambda_i = exp \int_{0}^{T} Tr D(\tau) d\tau \quad (3.18)$$

At time t_1 the multiplicity of the roots increases, which physically corresponds to the degenerate energy levels of the system, but the trace of the matrix under the integral remains constant. A similar formula has been found for the determinant of the Polish philosopher and mathematician Vronsky, which is useful for finding the derivative of a vector function in a system of linear differential equations.

The above mathematical formulae reveal the physical meaning of phase space and the trace of the matrix in the theory of ordinary differential equations. The adiabatic process and the constancy of entropy play an important role. There are other analogies to phase space $W \rightarrow \Gamma$ and entropy $Tr(D(\tau)) \rightarrow S$.

Note that the matrices are infinite and reduced to the Jordan form, the sigmoid function in question turns on/off the multiplicity of the roots and provides a transformation of the matrix just mentioned.

3.5 Fluctuations

For the $\forall$ formula (2.1) there is a whole class of physical subsystems: ideal gases, solutions, vibrations of the crystal lattice with quasi-particles - phonons, and so on, where the above approaches are implemented.

What is the quality indicator of the triple q or of the deviations from the stated principles? To answer this question, let's look at the whole system, including the subsystem under consideration and the fluctuations.

Taking into account the above arguments and the formula, it is necessary to find a suitable interpretation for the quality indicator of the triple q, the upper limit of which asymptotically approaches the fraction asymptotically approaches by some function with normal distribution as the entropy increases. (In thermodynamics, the number of molecules is of the order of the number/Avogadro constant - the number of particles contained in one mole of any substance $6.022 \dot{1} 0^{23}$). Most of the well-known examples of finding triples of numbers using supercomputers and distributed computing networks are so-called "good triples" [11,14] with $q > 1$ greater than one ($q_{max} \sim 1.4$), but these more athletic than scientific records are on the order of one ten-thousandth of the Avogadro constant.

Assuming an upper limit of q_{max} for the reference value of the three numbers A, B, C, for large N, it is more convenient to replace the word "quality" by a more appropriate "defect" according to the formula of the fluctuation from the mean, namely $\Delta q = \frac{\Delta S}{S_0}$. For an isolated subsystem, the very statement about fluctuations in energy, entropy, temperature, number of particles and other thermodynamic parameters is not appropriate - one can only talk about fluctuations

by considering the system as a whole. Let us define the equilibrium entropy of a body $S(E, V)$ as a function of its (average) energy and volume. By fluctuation of entropy we mean the change of the function $S(E, V)$, formally considered as a function of the exact (fluctuating) values of energy and volume. It is known that the probability of fluctuation in the system is proportional to the exponential deviation of the entropy:

$$w = e^{\Delta S} \qquad (3.19)$$

This is the entropy of the whole system.

Let's start by finding the mean square of the fluctuation in the number of particles of an ordinary ideal gas in a relatively small volume isolated in the gas. Assuming a uniform distribution of the gas molecules over the whole volume V_0 and the conditions for the smallness of the subsystem under consideration in comparison with the whole system $\frac{V}{V_0} \ll 1$, the square of the fluctuation of the number of particles in the subsystem (points in the manifold of the phase space 1^n) is determined by the formulae [9 Formula 113.1]:

$$\left\langle (\Delta N)^2 \right\rangle = N; \dots \frac{(\langle (\Delta N)^2 \rangle)^{1/2}}{N} = \frac{1}{\sqrt{N}} \qquad (3.20)$$

The relative fluctuation of the number of particles is equal to the inverse square root of the average number of particles. Similar formulas are known from textbooks of mathematical statistics [10]. Based on the average square of the fraction in a given volume of gas, a Gaussian probability distribution of the fluctuation of the number of particles is found:

$$w(N)dN = \frac{1}{\sqrt{2\pi\bar{N}}} exp \left(-\frac{(N - \bar{N})^2}{2\bar{N}} \right) dN \qquad (3.21)$$

For small N it is more appropriate to use the Poisson formula, known from mathematical statistics [10].

$$w_n = \frac{\bar{N}^N e^{-\bar{N}}}{N!} \qquad (3.22)$$

In the case we are interested in (assigned volume) $V \ll V_0$ (initial volume), the number of particles in the assigned volume is assumed to be small compared to the total number of particles in the gas, although it may differ significantly from its average value. Taking into account the *Stirling formula* $n! = \sqrt{2\pi n} \left(\frac{n}{e} \right)^n$, the formula (3.22) changes to the formula (3.21) with small N.

Regarding the question of entropy fluctuations, we note the following. According to Chebyshev's theorem [10, Chapter 9.3] and the additivity property of entropy, the Gaussian probability distribution (3.20 -3.22) can be applied to the fluctuation of entropy $\Delta S = \sqrt{S_0}$ and the upper bound of Δq_{max}. In fact, the number of particles in a gas and the entropy are additive quantities and *integrals of motion*, so they must be described by similar formulas. It is enough to imagine an experiment in which the initial container is divided into N small equivalent containers, where N is a rather large number, but obviously much smaller than the Avogadro number and the number of molecules in the system. Then, instead of the average number of particles, the entropy value can be substituted in the above formulae for estimating the fluctuation.

The question of the density of the distribution of prime numbers remains open in science. At the same time, you can enter a function that limits Δq from above.

$$\Delta q_{ABC_{max}} = \frac{(\langle (\Delta S)^2 \rangle)^{1/2}}{S} = \frac{1}{\sqrt{S_0}}$$
$$\approx \sqrt{\frac{L}{(ln(Rad(ABC)))}} \qquad (3.23)$$

Since the fluctuations of entropy obey the law of normal distribution and (3.23), the same is true for $\Delta q_{ABC_{max}}$.

To estimate the upper bound of the maximum deviation $\Delta q_{ABC_{max}}$, we can take into account the discrete nature of the phase space and the principle of equal probability of microstates, as well as the normalisation con-

dition (2.8). The maximum deviation of $\Delta q_{ABC_{max}}$ corresponds to the lowest probability value. In turn, the probability of such a microstate is determined only by the values on the main diagonal of the static matrix. In total there are L such elements, the smallest integer being one, so we have $w_{min} = \frac{1}{L}$, which corresponds to the law of normal distribution:

$$\frac{1}{L} = \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{\Delta q_{max}^2}{2\sigma^2}\right) \qquad (3.24)$$

Where the standard deviation of σ is determined by

$$\sigma = \sqrt{\frac{L}{(ln(Rad(ABC)))}} \qquad (3.25)$$

Note that $L \leq \pi(Rad(ABC))$ *Riemann functions* π. This statement simply says that the length of the chain of different primes in the formula (2.1) does not exceed the total number of primes not exceeding $Rad(ABC)$, determined by the Riemann function of the radical of the product A, B, C. Let us explain that the Riemann function $\pi(x)$ is defined as the number of prime numbers that do not exceed the integer number x. Chebyshev proved in 1851-1852 that if there is a limit, it must be equal to one:

$$lim_{x\to\infty}\frac{\pi(x)}{ln(x)/x} = 1 \qquad (3.26)$$

.

Which leads to the conclusion: $\frac{L}{ln(Rad(ABC))} \leq \frac{1}{Rad(ABC)}$.
This ratio is more accurate the higher the value of Rad(ABC). Solving the equation (3.24) and assuming that the expression under the radical sign is positive, we find

$$\Delta q_{ABC_{max}} = \sigma\sqrt{2ln\frac{L}{\sqrt{2\pi}\sigma}} \qquad (3.27)$$

The result of the algebraic transformations turns out to be cumbersome in the end, but if we take into account the ratio between the length of the chain of prime numbers and the value of the radical Rad(ABC)), small terms can be neglected (see below an alternative way of estimating $\Delta q_{ABC_{max}}$).

This is known from mathematical statistics courses:

$$|\Delta q_{ABC_{max}} < \epsilon| \leq 2\Phi\left(\frac{\epsilon}{\sigma}\right)\dots\Phi(x) =$$
$$\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{z^2}{2}}dz \qquad (3.28)$$

For any given ϵ, there are only a finite number of triples of numbers for which the upper value of the defect index: $\Delta q_{ABC_{max}} < x + \epsilon$ (do not confuse the arbitrary parameter ϵ with the energy of a single molecule, denoted by the same letter above!) Here x is the mathematical expectation Δq_{ABC}, depending on the distribution function of prime numbers, which is not known to modern science. But it can definitely be argued that there is a upper-limiting function $\Delta q_{ABC_{max}}$, the probability density distribution of which is subject to the law of normal distribution. It is easy to show that this conclusion is equivalent to another representation of the defect q_{max}, $max(A, B, C) < K(\varepsilon)rad(A, B, C)^{1+\varepsilon}$, where $K(\epsilon)$ is a certain constant that depends only on ϵ.

An alternative way of estimating $\Delta q_{ABC_{max}}$ follows from *The Six Sigma Rule*, which applies to the normal distribution in mathematical statistics (0.5 errors per 1 million). The minimum probability in the formula (3.24) is inversely proportional to the length of the chain L. A chain of two million of the product of prime numbers is so large that it exceeds the number of atoms in the universe (in the relation 3.26, you can replace the variables $y = lnx$ and make sure that the deviation $q_{ABC_{max}} \approx 3\sigma$ is characterised by the appearance in L of a chain (2.1) of prime numbers $\approx 10^{434294}$).

As a result, the discreteness factor of phase space due to the Heisenberg Uncertainty Principle will kick in earlier than the Six Sigma rule.
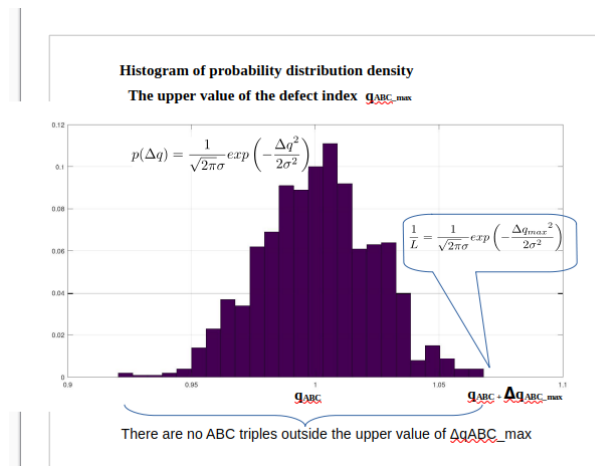
Figure 1: *This histogram illustrates the discrete nature of the phase space for the three ABC numbers under consideration. In the case of $q_{ABC} > q_{ABC} + \Delta q_{ABC_{max}}$ there is not a single triple that satisfies the ABC conjecture.*

The result obtained can be compared with the ABC@home project, a histogram of the quality of triples of q from the number of decimal places in ABC numbers. Figure 2.2 The ABC@home project finds all triples of ABC numbers for a given upper bound on the "quality" of qABC and the number of decimal places.
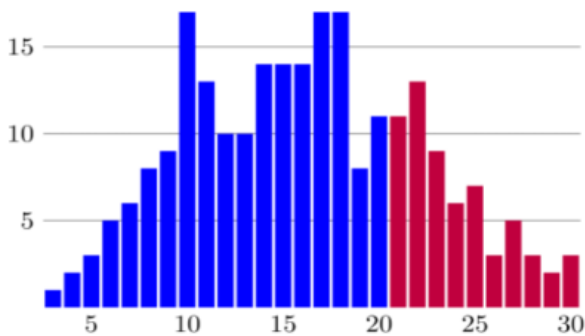
Figure 2: *The ABC@home project finds all triples of ABC numbers for a given upper bound on the "quality" of q_{ABC} and the number of decimals.*

The ABC@home project says *"The list of so-called good triples, consisting of 20 digits, is already complete. For example, the graph shows that there are 11 good triples in which c contains 20 digits. If new good triples are found, only the red part of the graph above will increase. (already 20 digits)"*. In fact, the project applies a logarithmic scale to the normal distribution

$q_{ABC_{max}} = 1 + \Delta q_{ABC_{max}}$ and then a quantitative method for dividing a set of ranked data into equal subsections (here- *deciles*), but a characteristic type of normal distribution (see above the Maxwell distribution (2.14) and 1). Instead of using supercomputers and distributed computing resources, the author simply created a 10,000 unit sample of the normal distribution with three lines of code and got a similar result in a few seconds in the cross-platform application GnuOctave (an analogue of MathLab under the GnuGPL licence) on a not-so-new desktop computer. Since the number of particles in a gas and entropy are additive quantities, the formula (2.23 -2.24) is also suitable for estimating entropy fluctuations. To do this, it is sufficient to imagine an experiment in which the initial container is divided into N equivalent containers (in the case of a mixture of gases, this operation should be carried out without separating the molecules of different gases, which would lead to a change in entropy), where N is a sufficiently large number, and at the same time much smaller than the Avogadro number / number of molecules in the system. Instead of the average number of particles, the entropy value must be substituted. .

4 Prime numbers as the metalanguage of the Universe

The conclusion of the canonical Gibbs distribution was made on the basis of an equally probable distribution over all microcanonical states of the system, the probability of finding a complex system 1-2 equal to the product of the probabilities of its subsystems 1 and 2: $\rho_{12} = \rho_1 \cdot \rho_2$, which makes it convenient to work with logarithms and allows one to operate with additive variables. Additive functions for homogeneous quantities are the basis for a set of natural numbers, for countable sets and for the production of

arbitrary measurements = > measurability of distances, introduction of the concepts of metrics and measures. The properties of the microcosm determine the properties of the macrocosm = > The operation of multiplication is primary to the operation of addition/subtraction. From the above positions it is possible to explain the axioms of arithmetic introduced by Giuseppe Peano, as well as the principle of mathematical induction, widely practised in mathematical proofs.

4.1 Summing up

The conclusions drawn can be generalised to a number of subsystems and apply equally to mathematics (foundations of probability theory, logic) and physics. Prime numbers as the metalanguage of the Universe (see Table 1 above).

It is logical to determine the logarithm of the probability distribution and obtain an additive value - entropy. The possibility of assigning a subsystem as part of a complex system and the possibility of defining a function allowed the architect of the universe (if there is one, despite the scepticism of atheists) to make the world manageable and knowable, to reduce complex phenomena to simple ones.

The association of prime numbers with logarithmic functions seems perfectly natural. In the course of evolution, the hearing, sight and touch of humans and animals have adapted to the logarithmic law, which allows them to perceive the environment adequately, organically continuing the laws of nature, is a consequence of the law of statistical mechanics and quantum physics, the probability theory.

To illustrate the manifestations of the properties of the microcosm, it is enough to refer to Heisenberg's uncertainty principle $\Delta p \Delta q \geqslant \hbar/2$, which manifests itself in Fourier transforms, in spectra and in signals: it is impossible to limit the spectrum and the duration of the signal at the same time.

The properties of polynomial rings and linear operators are derived from the properties of prime numbers. Due to the additive properties of entropy, it is possible to reduce the variation of easily measurable quantities: number of particles, energy, temperature, pressure, as the sample increases, and to describe them in simple linear relations. In this way, the relationship between the microcosm and the macrocosm is realised. This relationship obeys the laws of mathematical statistics (statistical physics). It is noteworthy that literally from the very first classes, children study phenomena based on quantum nature, the fundamental laws of our universe: symmetry, uniformity of space, isotropy, etc. To better understand the author's surprise, imagine a world without prime numbers, where each number has as many divisors as possible, where it is impossible to isolate a part from the whole, where each phenomenon is irreducible to simpler ones. . .

To summarise the above, it can be argued that the absence of prime numbers would change our universe beyond recognition. Would it be possible for a sane person to live in a world without prime numbers?

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